More on Geometrical Constructions of a Tangent to a Circle with a Straightedge Only

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Abstract

In geometrical constructions using a straightedge only the drawing of a tangent to a given circle without indication of the location of its center is a basic construction for solving many problems in this field. The correctness of this construction is usually based on concepts from Projective and Synthetic Geometry (polar, harmonic range etc.) which is not widely known among the undergraduate students and even among the teachers. In the article we give two elementary proofs of the correctness of such a construction: the first using analytical geometry and trigonometry and the second using the Euclidean geometry only. In addition, we shall discuss the advantages of using dynamic software for illustrating the construction of tangents.

1. Introduction

Since early history geometrical constructions using a straightedge and a compass have occupied mathematicians, and in particular geometricians. In accordance with the knowledge and skill acquired up to any particular period, there was progress in the ability to deal with geometrical constructions with a higher level of difficulty. These levels of difficulty required multifaceted reasoning, creativity and finding unorthodox methods of construction in order to carry out the tasks.

There are several reasons why geometric constructions are considered important [1]. First, geometric constructions are a part of the mathematical heritage and have been a popular part of mathematics throughout history. This popularity may possibly be attributed to the ancient Greeks' introduction of the following four famous geometric constructions which have been proven impossible to construct many hundreds of years later.

Naturally, delving deeper in order to find the method of solution, while combining mathematical knowledge and the capabilities of the drawing tools at the given conditions, contributes to the development of thought and every achieved special construction can be a basis for a more complex construction.

Though in classical construction problems it was customary to use two tools – a straightedge without gradations (unmarked ruler) and a compass, it is known that it is possible to make do with much less. In this context one should quote the findings of some important developers of geometry for modern times.

In 1797 the Italian geometrician Lorenzo Mascheroni has shown that any problem that can be constructed using a straightedge and a compass can be constructed using a compass only. To show this Mascheroni has proven that it is possible to add and subtract lengths of segments using a compass, and also to divide lengths by other lengths (straight lines cannot be constructed but it is possible to find two points that define the straight line).

Some 30 years later the Poncelet-Steiner theorem has stated that all the constructions that can be made using a straightedge and a compass can be carried out using a straightedge only, provided that a circle is given with the location of its center.

The present paper shall present the construction of tangents to a given circle without indication of the location of its center, using a straightedge only.

As will be seen, the construction of tangents using this method is simple even in comparison to the construction of tangents using a straightedge and a compass.

Up to now the proof of the method for constructing tangents using a straightedge only was based on special tools and properties from Projective and Synthetic geometry [2], [3, p.381, problem#10], [4]. The present paper presents the proofs that combine various fields in Geometry: Euclidean geometry, analytical geometry and trigonometry.

In the following we shall also present the construction problems that implement the method of construction of tangents to a circle using a straightedge only.

Finally, we discuss construction of tangents to circle by means of D.G.S. - Dynamic Geometric Software (for example, GeoGebra) in order to lay the importance of integrating technology in the process of teaching and learning on a firm basis.

It is a well-established today that DGS has opened new frontiers by linking informal argumentation with formal proof [5], [6]. The introduction of DGS (such as Geometers Sketchpad® and, recently, GeoGebra) into classrooms creates a challenge to the praxis of theorem acquisition and deductive proof in the study and teaching of Euclidean geometry. Students/learners can experiment using different dragging modalities on geometrical objects that they construct, and consequently they can infer properties, generalities, and conjectures about the geometrical artifacts. In fact, much earlier David Tall [7] mentioned in this sense the notion of `advance organizer` by Ausubel [8] and his own notion of `generic organizer`. Both notions are related and connected to the use of DGS. In case of Ausubel's advance organizer is: "Introductory material presented in advance of, and at a higher level of generality, inclusiveness, and abstraction than the learning task itself, and explicitly related both to existing ideas in cognitive structure and to the learning task itself ... i.e. bridging the gap between what the learner already knows and what he need to know to learn the material more expeditiously." [7]. Then, in case of Tall's generic organizer, it is defined "to be a microworld which enables the learner to manipulate examples of a specific mathematical concept or a related system of concepts. The term "generic" means that the learner's attention is directed at certain aspects of the examples which embody the more abstract concept." [7].

Using the operations of dragging a certain geometrical object and its influence on the obtained construction, a visual demonstration is presented which illustrates the method of construction. Because such non-deductive methods of investigation rely on experimentation, and intuitive and inductive reasoning [9], they provide more meaningful contexts for teaching and learning geometry with DGS; more so than the classical approach of proof. However, the students might become convinced that their conjecture will always be true [10], hence, because of the inductive nature of the DGS, we entitle this process "semi proof." As result of the employment of DGS, the experimental-theoretical gap that exists in the acquisition and justification of geometrical knowledge becomes an important pedagogical and epistemological concern [11].

2. Constructing tangents to a circle using a straightedge only

From a certain point A outside the circle whose center is not given, construct the tangents to the circle.

2.1 Description of the construction

From the point A using the straightedge, construct 3 secants of the circle, which intersect it at the points B, C, K, L, D, E as shown on figure 1.

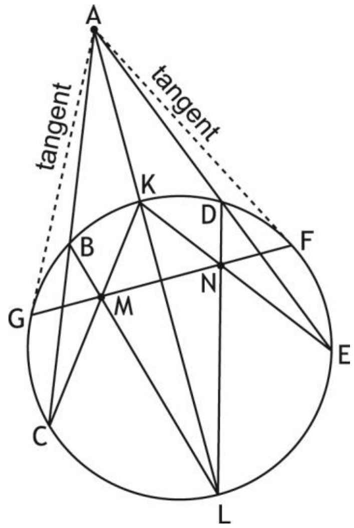

Figure 1

Draw the chords BL, KC, KE, DL. Denote by M and N the points of intersection of the straight lines KC and LB and the straight lines KE and DL, respectively.
Draw a straight line through the points M and N. This line intersects the circle at the points F and G. AF and AG are the tangents to the circle.

3. An example of construction implementing the construction of tangents using a straightedge only

Construct tangents to the circle from a given point outside a circle whose center is not given, and subsequently construct some triangle whose perimeter is equal to the sum of the lengths of the tangents.

3.1 Description of the construction and the proof

The description of the construction of the tangents AB and AC has been presented in the previous task. After constructing the tangents select a point D anywhere on the tangent AB, as shown in figure 2. From this point construct the tangent DE to the given circle using the same method as was used for the tangents AB and AC.
The continuation of the tangent DE intersects the tangent AC at the point F.
The perimeter of the triangle ∆ABC is equal to the sum of the lengths of the tangents AB and AC, since DB = DE and FC = FE.

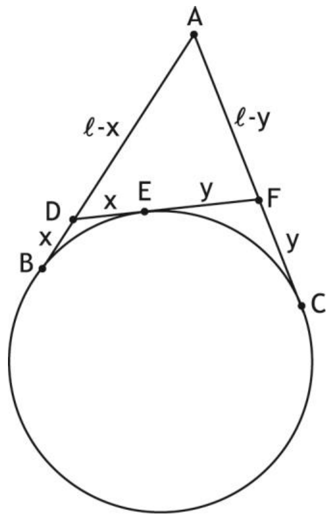

Figure 2

It is evident that the construction of tangents to the circle using a straightedge only is surprisingly simple, especially since it does not require the location of the center of the circle, any diameter, a segment with its midpoint or any other means as a precondition.
However, the proof of the correctness of the construction is complex and is only known in the version utilizing special tools and properties from Projective and Synthetic geometry.
The new two proofs are presented, that combine various methods and techniques from Euclidean geometry, analytical geometry and trigonometry.

4. Proof of the construction of the tangents

4.1 Stage A – Proving the tangent

Proof by trigonometry and analytic geometry

In a system of coordinates a circle is given with the unit radius, whose center is at the origin as shown in figure 3.
From a point N outside the circle draw two secants: NAB, where AB is a chord in the circle and NDC, where CD is a diameter in the circle. The continuations of the chords AD and CB intersect at the point P outside the circle.
The chords AC and BD intersect at the point Q. Connect the points P and Q by a straight line. The straight line intersects the circle at point M and the diameter at point K.

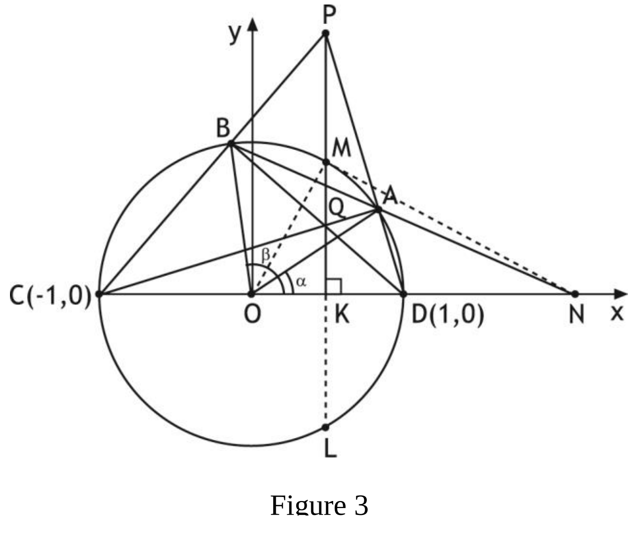

Figure 3

It remains to prove that the straight line MN is a tangent to the circle.
To prove this, the calculations shall show that the product of the slopes of the straight lines OM and MN is (-1).

We denote $\angle AOD = \alpha$, $\angle BON = \beta$.
In accordance with the notation and the selected system of coordinates we obtain the coordinates of the points A ($\cos\alpha$, $\sin\alpha$), B ($\cos\beta$, $\sin\beta$), C (-1, 0), D (1, 0).
From the slopes $m_{AD} = \dfrac{\sin\alpha}{\cos\alpha - 1}$, $m_{BC} = \dfrac{\sin\beta}{\cos\beta + 1}$ and from the coordinates of the points C and D we find the equation of the lines AD and BC, and by comparing them and using trigonometric identities, we obtain the x-coordinate of the point P:

$$x_P = \frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$

We have PK $\perp$ CD because the segments CA, DB and PK are the 3 altitudes of the triangle ΔCPD (inscribed angle resting on a diameter). Hence it follows that $x_P = x_M$.

Since the point M is located on the circle, we have $y_M = \dfrac{\sqrt{\cos^2\frac{\alpha-\beta}{2} - \cos^2\frac{\alpha+\beta}{2}}}{\cos\frac{\alpha-\beta}{2}}$.

To find the x-coordinate of the point N we use the slope

$$m_{AB} = \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = -\frac{\cos\frac{\alpha+\beta}{2}}{\sin\frac{\alpha+\beta}{2}} = -\operatorname{ctg}\frac{\alpha+\beta}{2},$$

and together with the coordinates of the point A we find the equation of the straight line AB. After substituting y = 0, we obtain $x_N = \dfrac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}$.

From the coordinates of the points M and N we calculate the slopes

$$m_{OM} = \frac{y_M - 0}{x_M - 0} = \frac{\sqrt{\cos^2\frac{\alpha-\beta}{2} - \cos^2\frac{\alpha+\beta}{2}}}{\cos\frac{\alpha+\beta}{2}}$$

$$m_{MN} = \frac{y_M - 0}{x_M - x_N} = -\frac{\cos\frac{\alpha+\beta}{2}}{\sqrt{\cos^2\frac{\alpha-\beta}{2} - \cos^2\frac{\alpha+\beta}{2}}}.$$

Hence, $m_{OM} \cdot m_{MN} = -1$, and therefore MN is a tangent to the circle at point M.

4.2 Proof by Euclidean geometry

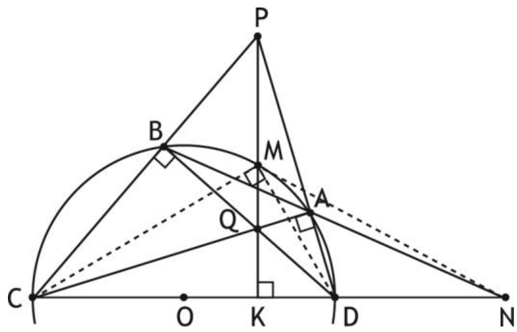

Figure 4

We denote
KD = x
DN = y
CD = 2R

According to Menelaus' Theorem ([12, Section 3.4]) there holds $\frac{CN}{DN}\cdot\frac{DA}{AP}\cdot\frac{BP}{BC}=1$ (*),

and according to Ceva's Theorem ([12, Section 1.2]) there holds $\frac{CK}{KD}\cdot\frac{DA}{AP}\cdot\frac{BP}{BC}=1$ (**).

From (*) and (**) we obtain that $\frac{CN}{DN}=\frac{CK}{KD}$.

Therefore, in accordance with the notation $\frac{2R+y}{y}=\frac{2R-x}{x}$, and from the properties of proportions we have

$$x=\frac{Ry}{R+y} \tag{1}$$

Using R and y we express the segments KN and KM: KN = x + y
By substituting the value of x from relation (1) we have

$$KN=\frac{y(2R+y)}{R+y} \tag{2}$$

The triangle ΔCMD is right-angled (inscribed angle resting on a diameter), and therefore, in accordance with the theorem of the altitude to the hypotenuse, we have

$MK^2 = CK\cdot KD = x\,(2R - x)$

By substituting the value of x from relation (1) we obtain

$$MK^2=\frac{R^2y(2R+y)}{(R+y)^2} \tag{3}$$

From the Pythagorean Theorem in the triangle ΔMKN, we have $MN^2 = MK^2 + KN^2$.
By substituting relations (2) and (3) in the last relation, we obtain, by using simple algebraic technique, that

$MN^2 = y\,(2R+y)$,

from where it follows that

$MN^2 = CN \cdot DN.$

Then from the theorem "the length of the tangent squared is equal to the product of the secant by its outer part", one can easily conclude that MN is a tangent.

The proof that the point M is the point of tangency was given by two methods. This testifies of the importance of gaining knowledge and skill in the different branches of mathematics. Sometimes employing tools from one field gives a quick and unique solution (or a proof) as opposed to the use of other tools, which do not give a solution at all, or a solution obtained by a cumbersome method. There are also cases when in order to obtain the solution, tools from different fields are combined. All these bring out the beauty of mathematics and increase the enjoyment, as was found in several studies (see, for instance, [13], [14], [15]).
It is important to note that there is a preference and a higher chance of successful handling of different tasks if one is acquainted with many theorems, such as Menelaus' Theorem, Ceva's Theorem, Pascal's Theorem and others, which not each student and even teacher knows.

4.3 Intermediate conclusion

Given a circle and its center, it is possible to construct the tangents to the circle from some point outside the circle using a straightedge only. From the chosen point construct 2 secants: one passing through the center of the circle and another at any other location. Connect the points of intersection and obtain two additional points of intersection: Q (internal) and P (external). The straight line PQ and its continuation intersect the circle at the points M and L, as can be seen on figure 5.

4.4 Stage B - proof that the location of the center of the circle is not required to construct the tangents

From some point N outside the circle draw three secants; NAB, NDC (which passes through the center of the circle) and NHG, as well as the tangents NM and NL, as shown on figure 5.

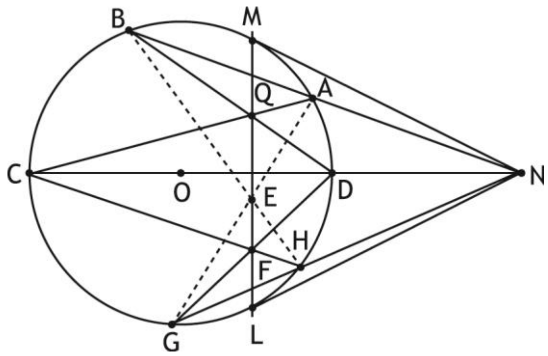

Figure 5

Draw all possible chords between the points of intersection of the secants and the circle. These chords intersect at the points Q, E and F.
Prove: the point E lies on the chord ML - the chord connecting the points of tangency.
Proof: in accordance with the proof of stage A. since CD is a diameter and ML $\perp$ CD, the points Q and F rest on the chord ML. It is to be proved that the third point of intersection also rests on the same chord.

To this end we use the Pascal theorem ([12, Section 3.8]), which concerns a general hexagon inscribed in a circle. The theorem states that the 3 points of intersection of the continuations of opposite (nonparallel) sides lie on a single straight line called the Pascal line.
The Pascal theorem holds for all kinds of inscribed hexagons including self-intersecting hexagon and this fact is used in the proof (figure 6).
In the present case we consider the self-intersecting hexagon HBDGACH that is inscribed in the circle and is composed of the segments HB, BD, DG, GA, AC, CH.
The sides HB and GA intersect at the point E, the opposite sides BD and AC intersect at the point Q, and opposite sides DG and CH intersect at the point F. According to the Pascal theorem the points E, Q and F lie on the same straight line. Therefore the point E is on the segment ML.

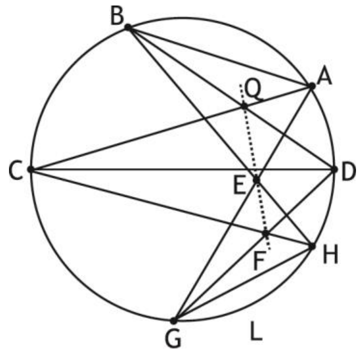

Figure 6

4.5 Conclusion from stage B

When three secants are drawn to a circle from an exterior point and all possible chords are drawn between their points of intersection, two points of intersection lying on the straight line connecting the points of intersection of the tangents to the circle are obtained, which issue from the point from which the secants issued. Using the straight line connecting the obtained two points of intersection we find the points of tangency.
Presenting solutions to problems by different methods, and requiring the students (whether in teacher-training courses or in school classes) to implement this method, enhances understanding and develops mathematical reasoning and creativity in the students. Solution of a problem by different methods adds options to the arsenal of tools of the student in dealing with the problem he or she faces in mathematics. Moreover, the process of solution employing different methods and techniques involves the application of rules of the subjects of Euclidean geometry and other fields , their integration and reinforcement both in the sense of their specific required skills, and, in some cases also skills from other mathematical fields such as algebraic skills, as a by-product. Using this method, the beauty and aesthetics intrinsic to mathematics are also expressed.
Concerning the method of proof, it is important to note that the solution of problems by combining fields in mathematics leads to creative and unorthodox solutions, which are the added value of acquaintance with different tools in mathematics. The proof of the construction of the tangents was made by using tools of mathematics from different fields, and at its end – on the Pascal theorem. This situation indicates the importance of acquaintance with “forgotten” theorems in geometry. Sometimes without them it is difficult to arrive at the final goal.

It is also important to note that proving a task using different methods, and in particular by combining different branches in mathematics, serves to illustrate the connections between them, reveals the beauty of mathematics and provides an insight to the fact that mathematics is a conjunction of intertwined fields. Silver et al. claim that "different solutions can facilitate connection of a problem at hand to different elements of knowledge with which a student may be familiar, thereby strengthening networks of related ideas" [16].
Geometry is a gold mine for multiple solution tasks and proofs by applying different methods within the specific topic of geometry and/or within other mathematical topics such as analytic geometry, trigonometry, etc. The multiple proofs foster creativity in mathematics and better comprehension of learners' mathematics.
The occupation with connections between different domains of mathematics, builds among students vision of mathematics as a linked science and not as a collection of discrete, isolated topics [17]. In most of the school textbooks, all over the world, problems in mathematics are organized by specific topics that are presented in the curriculum. In this case, the students know to which topic the problem is connected and hence assume that for each problem there is one and only one method to solve it [18].

5. Constructing a tangent to a circle through a given point on it

A natural question is raised of how one constructs a tangent to a circle at a point located on the circumference using a straightedge only? The method of construction of the tangent is much simpler
than in the previous case. To complete the picture we give the description of the construction, followed by the proof.

5.1 Description of the construction

Let A be the point on the circle through which a tangent shall be drawn. We select any four points on the circle B, C, D, E as shown on figure 7.

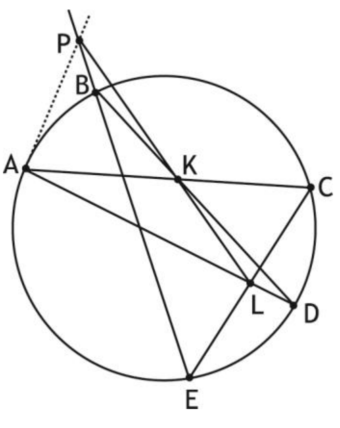

Figure 7

We construct the chords AC, AD, BD, BE and CE.
We denote by K the point of intersection of the chords AC and BD.
We denote by L the point of intersection of the chords AD and EC.

We connect the points K and L by a straight line. The continuation of the straight line LK intersects the continuation of the chord BE at the point P. The straight line connecting the points P and A is the sought tangent AP.

Note: if the straight lines KL and BE are parallel or almost parallel, we change slightly the position of point B and continue the construction as described.

5.2 Proof of the method of construction

We prepare a figure similar to figure 7 and pick another point F on the circumference of the arc $\overset{\frown}{AE}$ near the point A, as shown on figure 8.

With the point F we have obtained the self-intersecting hexagon ACEBDFA that is inscribed in the circle whose sides are AC, CE, EB, BD, DF, FA. The points of intersection of the opposing sides of the hexagon are as follows:

AC and BD intersect at K.

CE and DF intersect at L.

EB, and FA intersect at P.

According to the Pascal theorem, the points K, L and P lie on the same straight line.

When the point F is brought nearer to the point A, the points L, K, P remain on the same line and the line PAF approaches to the tangent at point A.

When the point F coincides with the point A, we obtain a degenerate hexagon, and the line PA is the tangent. The point P is obtained as the point of intersection of the continuations of the lines KL and BE, as described in the construction of the tangent.

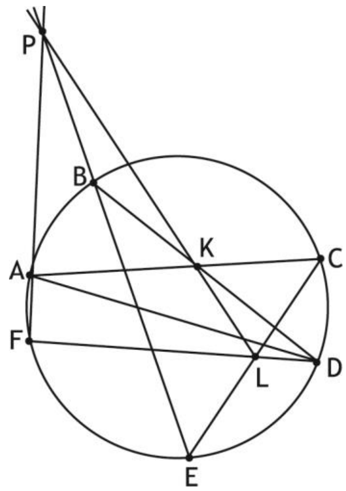

Figure 8

6. An example of construction implementing construction of the tangent to the circle at a point on the circle

Given is a circle with an inscribed equilateral triangle ΔABC. Find by construction using a straightedge only the location of the center of the circumscribing circle.

6.1 Description of the construction

In accordance with the method of construction of a tangent to a circle through a point on the circle, we construct tangents to the circle at the three vertices of the triangle ΔABC, as shown on figure 9.
The tangents through the vertices B and C intersect at point D.
The tangents through the vertices A and B intersect at point E.

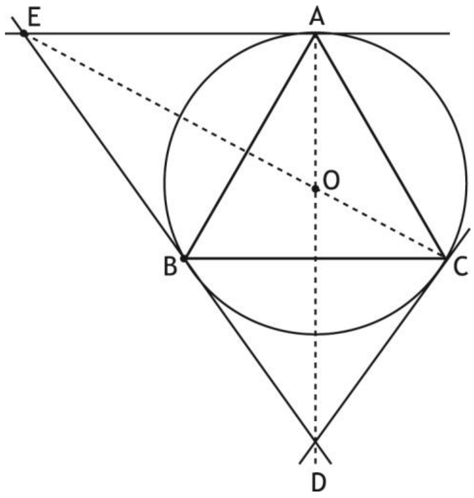

Figure 9

Hence, we obtaine two kites ABDC and AEBC. From the properties of a kite, the main diagonal of each kite is perpendicular to the sides of the triangle ΔABC, and bisects them. Hence, the diameter of the circle lies on each of the diagonals of the kite, and the point of intersection of the diagonals is the circumcenter (point O in the figure).

7. Utilizing computer technology in constructing tangents

The teaching of geometric constructions has great importance in the advancement of mathematical reasoning in general, and the geometrical reasoning in particular. Constructions are an important link for fixing the understanding of the role of the proof, which includes transition from an inductive process to the deductive process.
Research in education seeks methods for improving the quality of teaching and learning, and therefore it also focuses on integration of technology in teaching. The technological aid may attract the attention of the student through its ability to represent mathematical objects in a dynamic manner and provide the user with feedback in the solution of problems in different subjects. Learning which includes use of dynamic software allows the students to discover mathematical models, varied representations and relations between graphical descriptions, with reference to mathematical concepts ([19]). During learning with the aid of a technological tool the students can describe mathematical concepts better than in learning that does not include a technological tool. The learners achieve better understanding of these concepts and they gain access to high-level mathematical ideas ([20]).
Jones, Gutierrez, and Mariotti stated that it "provides a range of evidence that working with dynamic geometry software affords students possibilities of access to theoretical mathematics, something that can be particularly elusive with other pedagogical tools" [21]. Inductive exploration with the DGS, might lead students to form their own conjectures about the solution of the problem and then to deal with the deductive proof. This is in addition to the contribution of visualizing different graphical representations of concepts and other related situations to the

problem. As an implication for teaching, it may be recommended to mathematics teachers to let their students to initially cope with problem solving through work with this environment until they reach a solid conjecture for proving it deductively.

In order to use the GeoGebra software for constructing the tangents to the circle from point A, it is required to choose points C, E and G on the far arc of the circle. It is possible to drag each of these points on the arc of the circle and see that any such change in their location does not change the touch points M and N, but changes the location of the intersection points of chords H and I on the chord MN.
This construction is in accordance with the description of tangent construction using a straightedge only, as with the final stage of the proof of the method of construction using the Pascal theorem.
In other words, using the line connecting the points of intersection H and I of the pairs of chords, one can find the points of intersection M and N.

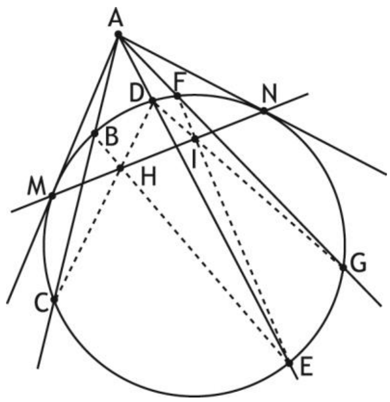

Figure 10

The point A was also selected so that it can be dragged anywhere, including inside the circle. It is clear that as this point is dragged, the length of the chord MN decreases. Gently we place the point A on the circumference, and we obtain a situation of a single tangent coinciding with the line MN. It is clear that by dragging the point A into the circle we do not obtain a tangent.

Another illustration by means of the computerized dynamic software was made for the case of construction of a tangent to a circle through a point located on it. In this case, points A, B, C, D, E are fixed points on the circumference, and only point F can be dragged on the circumference, as shown on figure 11. When dragging the point D or the point E on the arc of the circle, the position of points L, K and F changes, but these points remain on the same line, so the line PA remains the required tangent. When dragging the point F up to coincidence with the point A, the hexagon degenerates to a pentagon, and the dissection line PF becomes a tangent.

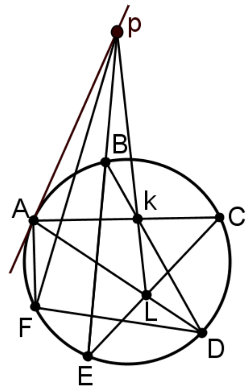

Figure 11

This demonstration together with the previous one constitute evidence to the fact that the proofs of the method of construction are correct, although they cannot replace proof, being only "semi-proofs".
Using the two links below it is possible to go directly to the construction of tangents using the GeoGebra (free software). In order to use it, the software first needs to be downloaded and installed. Link 1 is for constructing the tangent to the circle from a point outside the circle. Link 2 is for constructing the tangent at a point on the circle.

Link1 http://www.geogebratube.com/student/m65734

Link 2 http://www.geogebratube.com/student/m65735

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